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Best Multivariate Approximations by Trigonometric Polynomials with Frequencies from Hyperbolic Crosses*

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A characterization of smoothness properties which govern preassigned degrees of best approximations of multivariate periodic functions by trigonometric polynomials with the frequencies from so-called hyperbolic crosses is given in terms of new moduli of smoothness.

1. INTRODUCTION

One of the classical problems in univariate trigonometric polynomial (t.p.) approximation is the relationship between smoothness properties of a 2π-periodic function $f \in L_p(\mathbf{T})$, $1 \leqslant p \leqslant \infty$, $\mathbf{T} := [-\pi, \pi]$, and the degree of the error $E_n(f)_p$ of the best approximation by t.p. of order $< n$. The well-known direct and inverse theorem on this approximation states that $E_n(f)_p$ has a degree not greater than $n^{-\alpha}$, $\alpha > 0$, if and only if f possesses the Hölder fractional smoothness α, i.e., for some nonnegative integer $\beta < \alpha$ and natural number $r > \alpha - \beta$, the quasinorm

$$|f|_{H_p^\alpha} := \sup_{\delta > 0} \omega^r(f^{(\beta)}, \delta)_p / \delta^{\alpha - \beta}$$

is finite, where $\omega^r(f, \cdot)_p$ is the rth modulus of smoothness in $L_p(\mathbf{T})$ (cf., e.g., [11]). This theorem describes the quasinorm equivalence

$$|f|_{H_p^\alpha} \asymp |f|_{\mathscr{E}_p^\alpha}, \tag{1}$$

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where for $f \in L_p(\mathbf{T})$

$$|f|_{\mathscr{E}_p^\alpha} := \sup_{n \in \mathbf{N}} E_n(f)_p / n^\alpha.$$

Here and later, the statement $F_1 \asymp F_2$ means that $F_1 \ll F_2$ and $F_2 \ll F_1$, and the statement $F_1 \ll F_2$ means that $F_1 \leqslant CF_2$ with absolute constant C.

The situation is much more complicated in multivariate t.p. approximation on the d-dimensional torus $\mathbf{T}^d := [-\pi, \pi]^d$, because of the various possibilities for restricting the frequencies of t.p., and because multivariate functions can have very different and very complicated smoothness properties. Therefore, in multivariate t.p. approximations of smooth functions, we must first define a multivariate smoothness and then understand what frequency domain should be optimally selected for approximation of functions from the space of this common smoothness. The optimality of selection usually means that the corresponding approximation should give the degree of the widths or some other approximation characterizations of the unit ball of this space. Well-known function spaces of common smoothness are the isotropical and anisotropical Sobolev, Hölder, and Besov spaces. (Hölder spaces, although a subset of Besov spaces, are mentioned for their important place in t.p. approximations.) For these isotropical and anisotropical spaces the optimal frequency domains for t.p. approximation are d-balls and d-parallelepipeds, respectively. Moreover, the common smoothness of the Hölder and Besov spaces, which is defined on the basis of multivariate higher-order moduli of smoothness, and partial higher-order moduli of smoothness completely describes the smoothness properties governing the corresponding rate of the error of the best approximation by t.p. with frequencies from d-balls or d-parallelepipeds, in terms of quasinorm equivalence of type (1) of these spaces. This is a basic idea in the embedding theorems studied by Nikol'skii, Besov, and others. We refer the reader to [11] for a detailed description of this direction.

From the point of view of multivariate t.p. approximation the smoothness of the Hölder and Besov spaces and the frequency domains d-balls and d-parallelepipeds are rather simple, and the corresponding quasinorm equivalence theorems can be considered as direct generalizations of the univariate ones. More complicated anisotropical smoothness properties are the mixed Sobolev smoothness A of the space W_p^A of all functions f on $\mathbf{T}^d$ with $L_p(\mathbf{T}^d)$-bounded mixed derivatives in the sense of Weil $f^{(\alpha)}$ for all $\alpha \in A$, the mixed Hölder smoothness A and the mixed Besov smoothness A of the spaces H_p^A and $B_{p,q}^A$ of all functions f on $\mathbf{T}^d$ with the finite quasinorms $|f|_{H_p^\alpha}$ and $|f|_{B_{p,q}^\alpha}$, respectively, for all $\alpha \in A$, where A is a finite subset of $\mathbf{R}_+^d := \{x \in \mathbf{R}^d : x_j \geqslant 0\}$. The quasinorms $|f|_{H_p^\alpha}$ and $|f|_{B_{p,q}^\alpha}$ are defined in terms of mixed higher-order moduli of smoothness or, equivalently, mixed higher-order differences (cf., e.g., [5, 10, 13] for definitions).

The optimal frequency domain for these smoothness are so-called hyperbolic crosses (h.c.). Approximation by t.p. with frequencies from h.c. (h.c. approximations) of functions with mixed smoothness was pioneered by Babenko and the first important results were obtained by Mityagin [9] and Teljakovskii [12]. The reader can consult [5, 13] for detailed surveys on these and related problems. Because of the optimality of h.c. for mixed smoothness, it is of great interest to characterize the smoothness properties which guarantee a preassigned degree of the error in h.c. approximation. In order to formulate the setting of the problem, let us give some necessary definitions.

For a subset Γ of $\mathbf{Z}^d$, wet let $\mathscr{P}(\Gamma)$ denote the $L_p(\mathbf{T}^d)$-closure, $1 \leqslant p \leqslant \infty$, of the span of the harmonics e_k, $k \in \Gamma$, where $e_k(x) := e^{i\langle k, x\rangle}$, $x \in \mathbf{R}^d$. Let A be a finite subset of $\mathbf{R}^d_+$. We will treat the approximation by t.p. with frequencies from h.c. $\Gamma_A(t)$ which is defined by

$$\Gamma_A(t) := \left\{ k \in \mathbf{Z}^d : \prod_{j \in J_\alpha} |k_j|^{\alpha_j} < t, \alpha \in A \right\}, \qquad t > 0$$

where $J_\alpha := \{ j : 1 \leqslant j \leqslant d;\ \alpha_j \neq 0 \}$. We will use the abbreviated notation $\mathscr{P}_t^A := \mathscr{P}(\Gamma_A(t))$. We let

$$E_t^A(f)_p := \inf_{g \in \mathscr{P}_t^A} \|f - g\|_p \tag{2}$$

denote the error in the best $L_p(\mathbf{T}^d)$-approximation of $f \in L_p(\mathbf{T}^d)$ by elements from $\mathscr{P}_t^A$, where as usual $\|\cdot\|_p$ is the p-integral norm of $L_p(\mathbf{T}^d)$ with the change to the sup-norm when $p = \infty$. The most important case of h.c. approximation (2) is that where $\Gamma_A(t)$ is a finite subset of $\mathbf{Z}^d$ for each $t > 0$; i.e., the subspace $\mathscr{P}_t^A$ consists of all t.p. with frequencies from the h.c. $\Gamma_A(t)$. This occurs if and only if

$$a\varepsilon^j \in A, \qquad j = 1, ..., d, \quad \text{for some } a > 0,$$

where $\varepsilon^1 = (1, 0, ..., 0)$, $\varepsilon^2 = (0, 1, 0, ..., 0)$, ..., $\varepsilon^d = (0, ..., 0, 1)$ are the unit vectors in $\mathbf{R}^d$. However, we would like to emphasize that the results of the present paper will be stated without any requirement on finiteness of $\Gamma_A(t)$.

We are interested in a characterization of the smoothness properties of f which give a preassigned degree of $E_t^A(f)_p$. We let Φ denote the set of all functions $\varphi \in C([0, 1])$ such that $\varphi(x) > 0$ for $x > 0$, $\varphi(0) = 0$, and φ is nondecreasing on $[0, \tau]$ with some $0 < \tau \leqslant 1$. The degrees of $E_t^A(f)_p$, which we will consider, are of the form $\varphi(1/t)$ for $\varphi \in \Phi$ satisfying certain conditions of regularity (see Conditions (BS) and (Z_θ) below). Such typical degrees are functions $t^{-a}\log^b t$ with some $a>0$ and $b\in\mathbf{R}$ which appear in h.c. approximations of functions with finite mixed smoothness. Thus, from previous works concerning direct inequalities on h.c. approximations (cf. [5, 13] for details) we know that for $1<p<\infty$ the degree of $E_t^A(f)_p$ on the unit ball of W_p^A is t^{-1}, and the degree of $E_t^B(f)_p$ on the unit ball of H_p^A is $t^{-1}\log^{\nu/p^*} t$ with some nonnegative integer $\nu<d$, where $p^*:=\min(p,2)$, and B is a certain finite subset of $\mathbf{Z}_+^d$ constructed from A. A similar result is also valid for $B_{p,q}^A$ (see [5, 6]). Unfortunately, the smoothness of H_p^A and $B_{p,q}^A$ cannot characterize the smoothness properties for the corresponding degrees of the h.c. approximations $E_t^A(f)_p$ and $E_t^B(f)_p$.

Let us introduce spaces of functions with common degree of their error of h.c. approximation. If $\varphi\in\Phi$ and $0<q\leqslant\infty$, we let $\mathscr{E}_{p,q}^{A,\varphi}$ denote the space of all functions $f\in L_p(\mathbf{T}^d)$ such that the quasinorm

$$|f|_{\mathscr{E}_{p,q}^{A,\varphi}}:=\begin{cases}\left(\sum\limits_{n=0}^{\infty}\{E_{2^n}^A(f)_p/\varphi(2^{-n})\}^q\right)^{1/q}, & q<\infty\\ \sup\limits_{0\leqslant n<\infty}\{E_{2^n}^A(f)_p/\varphi(2^{-n})\}, & q=\infty,\end{cases}$$

is finite.

The aim of the present paper is to give a characterization of the smoothness properties of $\mathscr{E}_{p,q}^{A,\varphi}$, in terms of quasinorm equivalence theorems for Hölder-and Besov-type spaces. The most essential and difficult aspect of characterizing these smoothness properties is to introduce a suitable new modulus of smoothness in the definition of the corresponding Besov space. This problem was solved by DeVore *et al.* [4] for a single symmetric h.c., $1<p<\infty$, and $\varphi(t)=t^a$, $a>0$, by using new moduli of smoothness based on the $d-1$ dimensional integral of the convolution of higher-order mixed differences of functions with B-splines. Another characterization of the smoothness of $\mathscr{E}_{p,q}^{A,\varphi}$ for the case where $1<p<\infty$, $\varphi(t)=t^a$, $a>0$, and A is an arbitrary finite set, by introducing "moduli of smoothness" of functions with the help of convolutions of functions with certain distributions, was given in [7]. However, these last ones are not explicit nor are they moduli of smoothness in the traditional sense. In this paper we will introduce new moduli of smoothness for characterizing smoothness of $\mathscr{E}_{p,q}^{A,\varphi}$.

This paper is organized as follows. In Section 2 we introduce new moduli of smoothness and formulate the main results of the paper. In Section 3 we recall the Littlewood–Paley theorem and a modification of the Marcinkiewicz multiplier theorem which are the tools in the proofs of our results. We also obtain a series of auxiliary results to be used in these proofs. The proofs of the main results are given in Section 4.

2. MAIN RESULTS

We first give definitions of new moduli of smoothness. We motivate these definitions as follows. For a nonnegative integer r, the univariate difference operator Δ_h^r, $h \in \mathbf{T}$, is defined inductively by $\Delta_h^r := \Delta_h^1 \Delta_h^{r-1}$, starting from the operators

$$\Delta_h^0 f := f; \qquad \Delta_h^1 f := f(\cdot + h/2) - f(\cdot - h/2).$$

For $\beta \in \mathbf{Z}$, the univariate operator I_h^β, $h \in \mathbf{T}$, is defined in the same way, starting from the operators

$$I_h^0 f := f; \qquad I_h^1 f := h^{-1} g; \qquad I_h^{-1} f = h \frac{\partial f}{\partial x},$$

where g is the primitive with zero mean value of f, i.e.,

$$\frac{\partial g}{\partial x} = f, \qquad \int_{-\pi}^{\pi} g(x)\, dx = 0.$$

For a nonnegative integer r and $\beta \in \mathbf{Z}$, let us introduce the modulus of smoothness $\omega^{r,\beta}(f, \delta)_p$, $\delta > 0$, for $f \in L_p(\mathbf{T})$ by

$$\omega^{r,\beta}(f, \delta)_p := \sup_{|h| \leqslant \delta} \|\Delta_h^r I_h^\beta f\|_p.$$

If $\beta < 0$ and $s = -\beta$, then $\omega^{r,\beta}(f, \cdot)_p$ coincides with the classical rth modulus of smoothness $\omega^r(f^{(s)}, \cdot)_p$ of the sth derivative of f multiplied by δ^s. One can prove that for $f \in L_p(\mathbf{T})$, $1 \leqslant p \leqslant \infty$, $E_n(f)_p$ has order not greater than $n^{-\alpha}$ if and only if $\omega^{r,\beta}(f, \delta)_p \leqslant C\delta^\alpha$ for some integers β and r such that $r > \alpha + \beta > 0$ (in the case where $\beta \leqslant 0$ this is equivalent to the classical direct and inverse theorem), and, therefore, there hold the quasinorm equivalences

$$|f|_{H_p^\alpha} \asymp |f|_{\mathscr{E}_p^\alpha} \asymp \sup_{\delta > 0} \omega^{r,\beta}(f, \delta)_p / \delta^\alpha.$$

This means that the Hölder fractional smoothness α which governs the degree $n^{-\alpha}$ of $E_n(f)_p$ can be defined by finiteness of the quasinorm on the right-hand side of the last quasinorm equivalences. There hold similar quasinorm equivalences for Besov spaces. Based on this fact, our definition of new moduli of smoothness will be a generalization of $\omega^{r,\beta}(f, \cdot)_p$.

For $r \in \mathbf{Z}_+^d := \{k \in \mathbf{Z}^d : k_j \geqslant 0\}$, we let the multivariate mixed rth difference operator Δ_h^r, $h \in \mathbf{T}^d$, be defined by

$$\Delta_h^r f := \Delta_{h_1}^{r_1} \Delta_{h_2}^{r_2} \cdots \Delta_{h_d}^{r_d} f,$$

where the univariate operator $\Delta_{h_j}^{r_j}$ is applied to the variable x_j. For $\beta \in \mathbf{Z}^d$, the multivariate mixed operator I_h^β, $h \in \mathbf{T}^d$, is defined similarly. While the t.p. approximation is related to the traditional Hölder and Besov spaces, the direct generalization of $\omega^{r,\beta}(f, \delta)_p$ based on the mixed operator Δ_h^r and I_h^β does not give a desirable definition of smoothness for the h.c. approximation $E_t^A(f)_p$. The definition of the new moduli of smoothness in [4] is motivated by a similar argument. In our definition, we will first take the integral of $\Delta_h^r I_h^\beta f$ over the augmented $\delta V_\alpha(t) := V_\alpha(t) \backslash V_\alpha(t/2)$ of the hyperbolic neighborhood of the origin of coordinates $V_\alpha(t)$, $\alpha \in \mathbf{R}_+^d$, which is defined by

$$V_\alpha(t) = \left\{ h \in \mathbf{T}^d \colon h_j > 0, \prod_{j \in J_\alpha} h_j^{\alpha_j} < 2t \right\}, \qquad t > 0.$$

For a triple $\gamma = (\alpha, r, \beta) \in \mathbf{R}_+^d \times \mathbf{Z}_+^d \times \mathbf{Z}^d$, the operator D_t^γ is defined by

$$D_t^\gamma f := \int_{\delta V_\alpha(t)} \Delta_h^r I_h^\beta f \prod_{j \in J_\alpha} h_j^{-1} \, dh.$$

We define the modulus of smoothness $\Omega^\gamma(f, \delta)_p$, $\delta \geqslant 0$, by

$$\Omega^\gamma(f, \delta)_p := \sup_{t \leqslant \delta} \|D_t^\gamma f\|_p$$

and for a finite subset G of $\mathbf{R}_+^d \times \mathbf{Z}_+^d \times \mathbf{Z}^d$, the modulus of smoothness $\Omega^G(f, \delta)_p$ by

$$\Omega^G(f, \delta)_p := \sum_{\gamma \in G} \Omega^\gamma(f, \delta)_p.$$

We will later see that the modulus of smoothness $\Omega^G(f, \delta)_p$ is directly related to the h.c. $\Gamma_A(t)$, where $A = \{\alpha \colon (\alpha, r, \beta) \in G\}$. Note that for $\gamma = (\alpha, r, \beta)$ with $r_j \geqslant \beta_j$ the definition of the operator D_t^γ coincides with that which is given in [8]. We now define Besov spaces of common smoothness. If $\varphi \in \Phi$ and $0 < q \leqslant \infty$, we let $B_{p,q}^{G,\varphi}$ denote the Besov space of all functions $f \in L_p(\mathbf{T}^d)$ such that the quasinorm

$$|f|_{B_{p,q}^{G,\varphi}} := \begin{cases} \left(\sum_{n=0}^{\infty} \{\Omega^G(f, 2^{-n})_p / \varphi(2^{-n})\}^q \right)^{1/q}, & q < \infty \\ \sup_{0 \leqslant n < \infty} \{\Omega^G(f, 2^{-n})_p / \varphi(2^{-n})\}, & q = \infty, \end{cases}$$

is finite.

Let us compare moduli of smoothness $\Omega^G(f, \cdot)_p$ with partial moduli of smoothness. Denote by $\omega_j^s(f, \cdot)_p$ the sth partial modulus of smoothness of the variable x_j, (see, e.g., [11] for a definition). Then for any $\gamma = (\alpha, r, \beta)$ with $J_\alpha = J_r = J_\beta = \{j\}$, $\alpha_j = 1$ and $r_j - \beta_j = s$, there holds the inequality

$$\Omega^\gamma(f, \delta)_p \ll \omega_j^s(f, \delta)_p, \qquad 1 \leqslant p \leqslant \infty,$$

for $f \in L_p(\mathbf{T}^d)$. Moreover, by well-known methods, one can prove that $\omega_j^s(f, \cdot)_p$ can be substituted by such a $\Omega^\gamma(f, \cdot)_p$ in the corresponding quasinorm equivalence theorem for the anisotropical Hölder and Besov spaces.

We will require some conditions of regularity on φ. Namely, we say that $\varphi \in \Phi$ satisfies Condition (BS) if

$$\int_0^t \varphi(x) \frac{dx}{x} \ll \varphi(t),$$

and Condition (Z_θ), $\theta > 0$, if

$$\int_t^1 \varphi(x)\, x^{-\theta} \frac{dx}{x} \ll \varphi(t)\, t^{-\theta}.$$

We will need also some restriction on the set G for $\Omega^G(f, \cdot)_p$. We say that the set G satisfies Condition (R) if $J_\alpha = J_r = J_\beta$ and $1 < \beta_j < r_j$, $j \in J_\alpha$, for all $\gamma = (\alpha, r, \beta) \in G$. The moduli of smoothness $\Omega^\gamma(f, \cdot)_p$ with γ satisfying Condition (R) have a nice property. Namely, the multiplier coefficients of the related operators D_t^γ give a satisfactory rate of convergence for the application of the multiplier Marcinkiewicz theorem in the proofs of direct and inverse inequalities on h.c. approximations (Theorem 1).

Finally, let us introduce some quantities related to mixed smoothness and h.c. approximation. For G satisfying Condition (R), we define $\rho(G) := \min\{\rho(r - \beta, \alpha) : (\alpha, r, \beta) \in G\}$ and $v(G) := \max\{v(r - \beta, \alpha) : (\alpha, r, \beta) \in G, \rho(r - \beta, \alpha) = \rho(G)\}$, where $\rho(x, y) := \min\{x_j / y_j : j \in J_y\}$ and $v(x, y)$ denote the number of $j \in J_y$ such that $x_j / y_j = \rho(x, y)$. The number $\rho(G)$ can be interpreted as the "order" of $\Omega^G(f, \cdot)_p$. The number $v(G) - 1$ appears in degrees of h.c. approximations for classes of functions with mixed smoothness (cf., e.g., [5, 13]).

Denote by card Γ the cardinality of a set Γ. Let us now formulate direct and inverse inequalities on the h.c. approximation (2).

THEOREM 1. *Let* $1 < p < \infty$, $0 < q \leqslant \infty$, *and* A *be a finite subset of* $\mathbf{R}_+^d$. *Then for any* $\theta > 0$ *and any natural number* $v \leqslant \max\{\text{card } J_\alpha : \alpha \in A\}$, *we can construct a finite subset* G *of* $\mathbf{R}_+^d \times 2\mathbf{Z}_+^d \times \mathbf{Z}_+^d$ *such that*

(i) $A = \{\alpha : (\alpha, r, \beta) \in G\}$,

(ii) *G satisfies Condition (R),*

(iii) $\rho = \rho(G) \geqslant \theta$,

(iv) $\nu(G) = \nu$.

Moreover, if G is such a set and $f \in L_p(\mathbf{T}^d)$, then there holds the direct inequality of weak form

$$E^A_{2^n}(f)_p \ll \left(\sum_{m=n+1}^{\infty} \{\Omega^G(f, 2^{-m})_p\}^{p^*} \right)^{1/p^*} \tag{3}$$

for any nonnegative integer n whenever the right-hand side is finite. In addition we have the inverse inequality

$$\Omega^G(f, 2^{-n})_p \ll \left(\sum_{m=0}^{n} \{2^{-\rho(n-m)}(n-m)^{\nu-1} E^A_{2^m}(f)_p\}^{p^*} \right)^{1/p^*} \tag{4}$$

for any natural number n.

From Theorem 1 and a generalization of discrete Hardy inequalities we obtain the following Besov-type quasinorm equivalence theorem for characterizing the smoothness of $\mathscr{E}^{A,\varphi}_{p,q}$.

THEOREM 2. *Under the assumptions of Theorem* 1, *let $\varphi \in \Phi$ and φ satisfy Conditions (BS) and (Z_θ). Then for any finite subset G of $\mathbf{R}^d_+ \times \mathbf{2Z}^d_+ \times \mathbf{Z}^d_+$, satisfying Conditions (i)–(iii) in Theorem* 1, *we have*

$$\mathscr{E}^{A,\varphi}_{p,q} = B^{G,\varphi}_{p,q}.$$

Moreover for functions $f \in \mathscr{E}^{A,\varphi}_{p,q}$,

$$|f|_{\mathscr{E}^{A,\varphi}_{p,q}} \asymp |f|_{B^{G,\varphi}_{p,q}}.$$

Though the inequality (3) is of a form weaker than Jackson-type direct approximation inequalities, it is sufficient for establishing the quasinorm equivalence in Theorem 2. Moreover, the right-hand side of (3) is finite if f has a small fixed Hölder smoothness εA, i.e., $f \in H^{\varepsilon A}_p$ for arbitrary small $\varepsilon > 0$. From Theorem 1 it follows that we can construct a finite set G (with $\nu(G) = 1$) for which there holds the inequality (3) and the inequality

$$\Omega^G(f, 2^{-n})_p \ll \left(\sum_{m=0}^{n} \{2^{-\rho(n-m)} E^A_{2^m}(f)_p\}^{p^*} \right)^{1/p^*}. \tag{5}$$

This inequality may be considered as a multivariate modification of the univariate inequality proved by Stechkin for $p=2$ and by Timan and Timan for $1<p<\infty$ (see [15]). Some similar inequalities weaker than (3)–(5) were obtained in [4] for the best h.c. approximation, and moduli of smoothness considered by its authors.

Theorem 2 shows that for given p, q, φ, and A, different sets G satisfying the conditions of Theorem 2 determine the same space $B_{p,q}^{G,\varphi}$. Theorems 1 and 2 are restricted to the case where $1<p<\infty$, A is a finite set, and φ satisfies Conditions (BS) and (Z_θ). It seems significantly more difficult to treat the cases where $p=1$, ∞ or/and A is a infinite set and φ does not satisfy Conditions (BS) and (Z_θ).

The methods employed in the proofs of Theorems 1 and 2 are further development of those in the proofs of quasinorm equivalence theorems for the Hölder and Besov spaces H_p^A and $B_{p,q}^A$ of mixed smoothness (cf. [6, 10]) and also of those in [4] which rest on the Littlewood–Paley theorem and Marcinkiewicz multiplier theorem. In particular, our methods can be considered as a refinement of the methods of [4] which involve also some discrete Hardy inequalities.

The main results of the present paper were announced in [8].

3. AUXILIARY LEMMAS

For $s \in \mathbf{Z}_+^d$, we define the operator

$$\delta_s f = \sum_{k \in \square_s} \hat{f}(k)\, e_k$$

for $f \in L_p(\mathbf{T}^d)$, where $\square_s := \{k \in \mathbf{Z}^d : [2^{s_j-1}] \leqslant |k_j| < 2^{s_j}\}$ ($[a]$ denotes the integer part of a), and $\hat{f}(k)$ are the Fourier coefficients of f in the distributional sense.

Theorem 3 (Littlewood–Paley theorem, see e.g., [11]). *The following norm equivalence holds for $f \in L_p(\mathbf{T}^d)$, $1<p<\infty$,*

$$\|f\|_p \approx \left\| \left(\sum_{s \in \mathbf{Z}_+^d} |\delta_s f|^2 \right)^{1/2} \right\|_p.$$

Lemma 1 [7]. *Let $\{K(n)\}_{k=0}^{\infty}$ be a sequence of subsets $K(n) \subset \mathbf{Z}_+^d$ such that*

$$\mathbf{Z}_+^d \subset \bigcup_{n=0}^{\infty} K(n),$$

and let

$$f_n = \sum_{s \in K(n)} \delta_s f.$$

Then we have for $f \in L_p(\mathbf{T}^d)$, $1 < p < \infty$,

$$\|f\|_p \ll \left(\sum_{n=0}^{\infty} \|f_n\|_p^{p^*} \right)^{1/p^*}.$$

The Marcinkiewicz multiplier theorem (see, e.g., [11]) gives sufficient conditions on a sequence $\lambda = \{\lambda(k)\}_{k \in \mathbf{Z}^d}$ for the $L_p(\mathbf{T}^d)$-boundedness of the multiplier operator

$$\Lambda(f) := \sum_{k \in \mathbf{Z}^d} \lambda(k) \hat{f}(k) e_k.$$

In what follows, $\lambda(k)$ will be called the multiplier coefficients of Λ. We formulate a slight modification of the Marcinkiewicz multiplier theorem which immediately follows from this theorem and is more convenient for our applications. We let

$$Q_k := \{x \in \mathbf{R}^d : [2^{k_j - 1}] \leqslant |x_j| < 2^{k_j}\} \qquad \text{for} \quad k \in \mathbf{Z}^d_+$$

and

$$\mathscr{D}(e) := \prod_{j \in e} \frac{\partial}{\partial x_j} \qquad \text{for} \quad e \subset J := \{1, ..., d\}.$$

THEOREM 4. *Let* $\lambda = \{\lambda(k)\}_{k \in \mathbf{Z}^d}$ *be a sequence of values in* $\mathbf{Z}^d$ *of a function* $\lambda(x)$ *defined on* $\mathbf{R}^d$ *and satisfying the conditions*

$$\sup_{x \in \mathbf{R}^d} |\lambda(x)| \leqslant M,$$

and for any $s \in \mathbf{Z}^d_+$ *and* $e \subset J$

$$\sup_{x \in Q_s} |\mathscr{D}(e) \lambda(x)| \leqslant M \prod_{j \in e} 2^{-s_j}.$$

Then the multiplier operator Λ *with the multiplier coefficients* $\lambda(k)$ *is a bounded operator on* $L_p(\mathbf{T}^d)$, $1 < p < \infty$, *and for* $f \in L_p(\mathbf{T}^d)$

$$\|\Lambda(f)\|_p \leqslant CM \|f\|_p,$$

where C *depends only on* p *and* λ.

We shall employ Theorem 4 to establish some properties of the operator D_t^γ. We let $\mathbf{1} = (1, 1, ..., 1) \in \mathbf{R}^d$. For $\alpha \in \mathbf{R}_+^d$ and $\eta > 0$, we define

$$Z(\alpha, \eta) := \left\{ \cup \square_s \colon \prod_{j \in J_\alpha} s_j \neq 0, \eta - 1 \leqslant \langle \alpha, s \rangle < \eta \right\},$$

$$U(\alpha, \eta) := \{ x \in \mathbf{R}_+^d \colon \langle \alpha, x \rangle \geqslant \eta + c \} \qquad (c > 0).$$

LEMMA 2. *Let* $1 < p < \infty$ *and* $\gamma = (\alpha, r, \beta) \in \mathbf{R}_+^d \times \mathbf{Z}_+^d \times \mathbf{Z}_+^d$ *satisfy Condition* (R). *Then for any* $\xi, \eta > 0$ *and any* $f \in \mathscr{P}(Z(\alpha, \eta))$, *we have*

$$\|D_{2^{-\xi}}^\gamma f\|_p \ll \|f\|_p \begin{cases} 2^{-\rho(\xi - \eta)}(\xi - \eta + 1)^{v-1}, & \eta \leqslant \xi \\ 2^{-\rho'(\eta - \xi)}(\eta - \xi + 1)^{v'-1}, & \eta \geqslant \xi, \end{cases}$$

where $\rho = \rho(r - \beta, \alpha)$, $v = v(r - \beta, \alpha)$ *and* $\rho' = \rho(\beta - \mathbf{1}, \alpha)$, $v' = v(\beta - \mathbf{1}, \alpha)$.

To prove Lemma 2 we need the following:

LEMMA 3. *Let* $\alpha, \beta \in \mathbf{R}_+^d$ *with* $\alpha_j, \beta_j > 0$. *Then we have for any* $\xi \geqslant 0$

$$\int_{U(\alpha, \xi)} 2^{-\langle \beta, x \rangle}\, dx \approx 2^{-\rho\xi}(\xi + 1)^{v-1},$$

where $\rho = \rho(\beta, \alpha)$, $v = v(\beta, \alpha)$.

Proof. It follows from a result in [5] that

$$\int_{U(\alpha, \xi)} 2^{-\langle \beta, x \rangle}\, dx \approx 2^{-\omega\xi}(\xi + 1)^m,$$

where $\omega = \min\{\langle \beta, x \rangle : x \in U(\alpha, 1)\}$ and m is the dimension of the affine hull of the set $V = \{x \in U(\alpha, 1) : \langle \beta, x \rangle = \omega\}$. Therefore, to prove the lemma it is sufficient to show that $\rho = \omega$ and $v = m + 1$. Without loss of generality we can assume that $\rho = \beta_j / \alpha_j$ for $1 \leqslant j \leqslant v$. For $x^* = (1/\alpha_1, 0, ..., 0) \in U(\alpha, 1)$, we have $\langle \beta, x^* \rangle = \rho$. This means that $\rho \geqslant \omega$. Since $\beta - \rho\alpha \in \mathbf{R}_+^d$, we have $\langle \beta, x \rangle = \rho \langle \alpha, x \rangle + \langle \beta - \rho\alpha, x \rangle \geqslant \rho \langle \alpha, x \rangle \geqslant \rho$ for any $x \in U(\alpha, 1)$. This implies that $\omega \geqslant \rho$. Thus the equality $\rho = \omega$ has been proved. The equality $v = m + 1$ follows from the equality $V = \{x \in \mathbf{R}_+^d : \langle x, \mathbf{1} \rangle = 1, x_{j'} = 0, v < j' \leqslant d\}$. Lemma 3 is proved.

Proof of Lemma 2. We shall consider only the case $J_\alpha = J$, because the case $J_\alpha \neq J$ can be reduced to the case $J_\alpha = J$ for functions of d' variables where $d' = \operatorname{card} J_\alpha$. For the univariate operators Δ_h^1 and I_h^1, we have $\widehat{(\Delta_h^1 f)}(k) = (2i \sin kh/2)\, \hat{f}(k)$, $\widehat{(I_h^1 f)}(k) = (ikh)^{-1}\, \hat{f}(k)$ and $\widehat{(I_h^1 f)}(0) = 0$.

Hence, a simple computation shows that $\delta_h^r I_h^\beta$ is a multiplier operator of the form

$$\Delta_h^r I_h^\beta f = \sum_{k \in \mathbf{Z}^d} \mu_h(k)\, \hat{f}(k)\, e_k,$$

where for $x \in \mathbf{R}^d$

$$\mu_h(x) = \begin{cases} \prod_{j=1}^{n} (2i \sin x_j h_j/2)^{r_j}/(ih_j x_j)^{\beta_j}, & \prod_{j=1}^{d} x_j \neq 0 \\ 0, & \prod_{j=1}^{d} x_j = 0. \end{cases} \tag{6}$$

For $\xi, \eta > 0$, we define the multiplier operator $\Lambda_{\xi,\eta}$ by its multiplier coefficients $\lambda_{\xi,\eta}(k)$, $k \in \mathbf{Z}^d$, which are the values at k of the function

$$\lambda_{\xi,\eta}(x) := \begin{cases} \int_{\delta V_\alpha(2^{-\xi})} \mu_h(x) \prod_{j=1}^{d} h_j^{-1}\, dh_j, & x \in R(\alpha, \eta) \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

where $R(\alpha, \eta) := \{\cup Q_s : s \in \mathbf{N}^d, \eta - 1 \leqslant \langle \alpha, s \rangle < \eta\}$. For any $t, x \in \mathbf{R}$ such that $tx \neq 0$, we let $g_t(x) := (tx)^{-b} \sin^a(tx/2)$ $(a > b > 1)$. Then we have the estimates

$$|g_t(x)| \ll \min(|tx|^{a-b}, |tx|^{-b}), \tag{8}$$

$$|g_t'(x)| \ll x^{-1} \min(|tx|^{a-b}, |tx|^{1-b}). \tag{9}$$

We shall use (8)–(9) to apply Theorem 4 to $\Lambda_{\xi,\eta}$. By (6)–(8) we have for any $x \in Q_s \subset R(\alpha, \eta)$

$$|\lambda_{\xi\eta}(x)| \ll \int_{\delta V_\alpha(2^{-\xi})} \prod_{j=1}^{d} \min\{(h_j 2^{s_j})^{r_j - \beta_j}, (h_j 2^{s_j})^{-\beta_j}\} \frac{dh_j}{h_j}.$$

Putting $h_j = 2^{-y_j}$, $y = (y_1, ..., y_d)$ in the right-hand side of this inequality, we obtain

$$|\lambda_{\xi,\eta}(x)| \ll \int_{H(\alpha,\xi)} \prod_{j=1}^{d} \min\{2^{r_j'(s_j - y_j)}, 2^{-\beta_j(s_j - y_j)}\, dy\}, \tag{10}$$

where $H(\alpha, \xi) := \{y \in \mathbf{R}^d : \xi - 1 \leqslant \langle \alpha, y \rangle \leqslant \xi\}$, $r_j' = r_j - \beta_j$. If $\tau := \xi - \eta \geqslant 0$, putting $z_j = y_j - s_j + \xi_0$, $j = 1, ..., d$, $\xi_0 = 1/\langle \alpha, \mathbf{1} \rangle$ in the right-hand side of the last inequality, we get for any $x \in Q_s \subset R(\alpha, \eta)$

$$|\lambda_{\xi,\eta}(x)| \ll \int_{\langle \alpha, z \rangle \geqslant \tau} 2^{-\sum_{j=1}^{d} \max(r_j' z_j, -\beta_j z_j)}\, dz =: \mathscr{J}. \tag{11}$$

The integral $\mathcal{J}$ can be decomposed into a sum of the integrals

$$\mathcal{J} = \sum_{e \subset J} \mathcal{J}(e), \qquad \mathcal{J}(e) := \int 2^{\langle r^*, z \rangle} \, dz, \tag{12}$$

where the integral $\mathcal{J}(e)$ is taken over the set

$$\left\{ z \in \mathbf{R}^d_+ : \sum_{j \in e} \alpha_j z_j - \sum_{j \in e'} \alpha_j z_j \geqslant \Phi \right\}, \qquad e' := J \backslash e$$

for any $e \subset J$ and $r_j^* = r_j'$ for $j \in e$ and $r_j^* = -\beta_j$ for $j \in e'$. The integral $\mathcal{J}(e)$ can be estimated as

$$\mathcal{J}(e) \leqslant \int 2^{\langle r^*, z \rangle} \, dz = \mathcal{J}_1(e) \, \mathcal{J}_2(e),$$

where the integral is taken over the set $\{ z \in \mathbf{R}^d_+ : \sum_{j \in e} \alpha_j z_j \geqslant \tau \}$ and

$$\mathcal{J}_1(e) = \int_{z_j \geqslant 0} 2^{-\sum_{j \in e'} \beta_j z_j} \prod_{j \in e'} dz_j,$$

$$\mathcal{J}_2(e) := \int_{U(\alpha, e, \tau)} 2^{-\sum_{j \in e} r_j' z_j} \prod_{j \in e} dz_j,$$

with $U(\alpha, e, \tau) = \{ (z_j)_{j \in e} : \sum_{j \in e} \alpha_j z_j \geqslant \tau \}$. The integral $\mathcal{J}_1(e)$ is an absolute constant. The integral $\mathcal{J}_2(e)$ is estimated by Lemma 3. $\mathcal{J}_2(e) \approx 2^{-\rho(e)\tau} (\tau + 1)^{\nu(e) - 1}$ with $\rho(e) = \min\{ r_j'/\alpha_j : j \in e \} \geqslant \rho$ and $\nu(e) = \operatorname{card}\{ j \in e : r_j'/\alpha_j = \rho(e) \} \leqslant \nu$. Therefore, we have $\mathcal{J}(e) \ll 2^{-\rho\tau} (\tau + 1)^{\nu - 1}$. This and (11)–(12) imply that

$$\sup_{x \in \mathbf{R}^d} |\lambda_{\xi, \eta}(x)| \ll 2^{-\rho(\xi - \eta)} (\xi - \eta + 1)^{\nu - 1}, \qquad \xi \geqslant \eta. \tag{13}$$

If $\tau = \eta - \xi \geqslant 0$, putting $z_j = s_j - y_j - \xi_0$, $j = 1, ..., d$, in the right-hand side of (10), we get for any $x \in Q_s \subset R(\alpha, \eta)$

$$|\lambda_{\xi, \eta}| \ll \int_{\langle \alpha, z \rangle \geqslant \tau} 2^{-\sum_{j=1}^d \max(-r_j' z_j, \beta_j z_j)} \, dz.$$

Hence, similarly to (13) in the case $\xi \geqslant \eta$, we can prove that

$$\sup_{x \in \mathbf{R}^d} |\lambda_{\xi, \eta}(x)| \ll 2^{-\rho'(\eta - \xi)} (\eta - \xi + 1)^{\nu' - 1}, \qquad \xi \leqslant \eta. \tag{14}$$

Applying (9) to each variable, similarly to (10), we obtain for any $e \subset J$ and $x \subset Q_s \subset R(\alpha, \eta)$

$$|\mathscr{D}(e)\, \lambda_{\xi,\eta}(x)| \ll \prod_{j \in e} 2^{-s_j} \int_{H(\alpha,\xi)} \prod_{j=1}^{d} \min\{2^{r'_j(s_j-y_j)}, 2^{\beta'_j(s_j-y_j)}\}\, dy, \quad (15)$$

where $\beta'_j = \beta_j - 1$ for $j \in e$, and $= \beta_j$ for $j \notin e$. The integral in (15) can be estimated in the same way as the integral in (16). Thus we obtain the inequality

$$\sup_{x \in Q_s} |\mathscr{D}(e)\, \lambda_{\xi,\eta}(x)| \ll \prod_{j \in e} 2^{-s_j} \begin{cases} 2^{-\rho(\xi-\eta)}(\xi-\eta+1)^{\nu-1}, & \eta \leqslant \xi \\ 2^{-\rho'(\eta-\xi)}(\eta-\xi+1)^{\nu'-1}, & \eta \geqslant \xi. \end{cases} \quad (16)$$

Applying Theorem 4 to the multiplier operator $\Lambda_{\xi,\eta}$ satisfying Conditions (13)–(14) and (16), we arrive at the estimate

$$\|\Lambda_{\xi,\eta} f\|_p \ll \|f\|_p \begin{cases} 2^{-\rho(\xi-\eta)}(\xi-\eta+1)^{\nu-1}, & \eta \leqslant \xi \\ 2^{-\rho'(\eta-\xi)}(\eta-\xi+1)^{\nu'-1}, & \eta \geqslant \xi \end{cases}$$

for any $f \in L_p(\mathbf{T}^d)$. This implies the lemma, because $D^{\gamma}_{2^{-\xi}} f = \Lambda_{\xi,\eta} f$ for $f \in \mathscr{P}(Z(\alpha, \eta))$.

By a method similar to that in the proof of Theorem 2.6 in [4], we can prove the following

LEMMA 4. *Let* $1 < p < \infty$ *and* $\gamma = (\alpha, r, \beta) \in \mathbf{R}^d_+ \times 2\mathbf{Z}^d_+ \times \mathbf{Z}^d_+$ *satisfy Condition* (*R*). *Then for any* $\xi > 0$ *and any* $f \in \mathscr{P}(Z(\alpha, \xi))$,

$$\|f\|_p \ll \|D^{\gamma}_{2^{-\xi}} f\|_p.$$

If $0 < q \leqslant \infty$ and $\varphi \in \Phi$, we define the norm $\|a\|_{q,\varphi}$ for a sequence $a = \{a_k\}_{k \in \mathbf{Z}}$ as

$$\|a\|_{q,\varphi} := \begin{cases} \left(\sum_{k \in \mathbf{Z}} \{|a_k|/\varphi(2^{-k})\}^q \right)^{1/q}, & q < \infty \\ \sup_{k \in \mathbf{Z}} \{|a_k|/\varphi(2^{-k})\}, & q = \infty. \end{cases}$$

The following lemma is a generalization of the discrete Hardy inequality (see, e.g., [3, p. 27]).

LEMMA 5. *Let* $0 < q \leqslant \infty$, $\nu \in \mathbf{R}$, $\varphi \in \Phi$ *and* φ *satisfy the Conditions* (BS) *and* (Z_θ). *If the sequences* $a = \{a_k\}_{k \in \mathbf{Z}}$ *and* $b = \{b_k\}_{k \in \mathbf{Z}}$ *with* $a_k, b_k \geqslant 0$, *satisfy the condition*

$$b_k \leqslant M \left\{ \sum_{s \geqslant k} a_s + 2^{-\theta k} \sum_{s \leqslant k} 2^{\theta s} (k - s + 1)^\nu \, a_s \right\}, \tag{17}$$

then

$$\|b\|_{q, \varphi} \leqslant CM \, \|a\|_{q, \varphi} \tag{18}$$

with C *depending only on* q, ν, θ, φ.

Proof. We shall prove the case $1 \leqslant q < \infty$ of this lemma. The other cases can be proved with a slight modification. We will need a more convenient characterization of Conditions (BS) and (Z_θ) for our application. It was proved in [2] that Conditions (BS) and (Z_θ) on the approximation degree φ are equivalent to the inequality

$$\varphi(t) \, t^{-2\zeta} \ll \varphi(t')(t')^{-2\zeta}, \qquad t < t', \tag{19}$$

with some $0 < 2\zeta < 1$, and the inequality

$$\varphi(t) \, t^{-\delta} \ll \varphi(t')(t')^{-\delta}, \qquad t > t', \tag{20}$$

with some $0 < \delta < \rho$, respectively. From (20) it follows that φ satisfies Condition $(Z_{\theta'})$ with any $\delta < \theta' < \theta$. Let the sequences a and b satisfy Condition (17). If $\nu \neq 0$, then a and b also satisfy (17) with $M' = \lambda M$, $\delta < \theta' < \theta$, and $\nu' = 0$, where λ is some absolute constant. This allows us to treat only the case $\nu = 0$. For the sake of simplicity we assume that $M = 1$ in (17). We shall, for the moment, use the abbreviated notation $\gamma_n = 1/\varphi(2^{-n})$. Condition (17) gives

$$\|b\|_{q, \varphi} \leqslant \left(\sum_{s \in \mathbf{Z}} \gamma_s^q \left(\sum_{k \geqslant s} a_k + 2^{-\theta s} \sum_{k \leqslant s} 2^{\theta k} \, a_k \right)^q \right)^{1/q} =: \mathscr{J}. \tag{21}$$

We have

$$\mathscr{J} \leqslant \mathscr{J}_1 + \mathscr{J}_2, \tag{22}$$

where

$$\mathscr{J}_1^q := \sum_{s \in \mathbf{Z}} \left(\gamma_s 2^{-\theta s} \sum_{k \leqslant s} 2^{\theta k} a_k \right)^q, \qquad \mathscr{J}_2^q := \sum_{s \in \mathbf{Z}} \left(\gamma_s \sum_{k \geqslant s} a_k \right)^q.$$

We first estimate $\mathscr{J}_1$. The Hölder inequality gives

$$\sum_{k \leqslant s} 2^{\rho k} a_k \leqslant \left\{ \sum_{k \leqslant s} (2^{\varepsilon k} \gamma_k a_k)^q \right\}^{1/q} \left\{ \sum_{k \leqslant s} (2^{(\delta + \varepsilon) k} \gamma_k^{-1})^{q'} \right\}^{1/q'},$$

where $1/q + 1/q' = 1$, $\varepsilon = (\rho - \delta)/2 > 0$, and δ is in (20). We have by (20)

$$\left\{ \sum_{k \leqslant s} (2^{(\delta + \varepsilon) k} \gamma_k^{-1})^{q'} \right\}^{1/q'} \ll \gamma_s^{-1} 2^{\delta s} \left(\sum_{k \leqslant s} 2^{q' \varepsilon k} \right)^{1/q'} \ll \gamma_s^{-1} 2^{(\delta + \varepsilon) s}.$$

Therefore, $\mathscr{J}_1^q$ can be estimated as

$$\mathscr{J}_1^q \ll \sum_{s \in \mathbf{Z}} 2^{-q \varepsilon s} \sum_{k \leqslant s} (2^{\varepsilon k} \gamma_k a_k)^q$$
$$= \sum_{k \in \mathbf{Z}} (2^{\varepsilon k} \gamma_k a_k)^q \sum_{s \geqslant k} 2^{-q \varepsilon s} \ll \sum_{k \in \mathbf{Z}} (\gamma_k a_k)^q.$$

Thus, we have proved that

$$\mathscr{J}_1 \ll \|a\|_{q, \varphi}. \tag{23}$$

We next estimate $\mathscr{J}_2$. Again, the Hölder inequality gives

$$\sum_{k \geqslant s} a_k \leqslant \left\{ \sum_{k \geqslant s} (2^{-\zeta k} \gamma_k a_k)^q \right\}^{1/q} \left\{ \sum_{k \geqslant s} (2^{\zeta k} \gamma_k^{-1})^{q'} \right\}^{1/q'},$$

where ζ is in (19). We have by (19)

$$\left\{ \sum_{k \geqslant s} (2^{\zeta k} \gamma_k^{-1})^{q'} \right\}^{1/q'} \ll \gamma_s^{-1} 2^{2 \zeta s} \left(\sum_{k \geqslant s} 2^{-q' \zeta k} \right)^{1/q'} \ll \gamma_s^{-1} 2^{\zeta s}.$$

Hence, we obtain

$$\mathscr{J}_2^q \ll \sum_{s \in \mathbf{Z}} 2^{q \zeta s} \sum_{k \geqslant s} (2^{-\zeta k} \gamma_k a_k)^q$$
$$= \sum_{k \geqslant 0} (2^{-\zeta k} \gamma_k a_k)^q \sum_{s \leqslant k} 2^{q \zeta s} \ll \sum_{k \geqslant 0} (\gamma_k a_k)^q.$$

This means that $\mathscr{J}_2 \ll \|a\|_{q, \varphi}$. Combining the last inequality and (21)–(23) gives (18). The lemma is proved.

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. For any $\theta > 0$ and any natural number $v \leqslant d$, we will construct a subset G of $\mathbf{R}^d_+ \times 2\mathbf{Z}^d_+ \times \mathbf{R}^d_+$ satisfying Conditions (i)–(iv) in Theorem 1. Clearly, it is sufficient to do this for the case where $A = \{\alpha\}$ for some $\alpha \in \mathbf{R}^d_+$. Without loss of generality we can assume that $J_\alpha = J$ and $0 < \alpha_1 = \alpha_2 = \cdots = \alpha_s < \alpha_{s+1} \leqslant \cdots \leqslant \alpha_d$ for some $0 \leqslant s \leqslant d$. Let b be a natural number such that $1 < b < [\theta\alpha_1] + 1$ and $[\theta\alpha_1] + b + 1$ is an even number. Then, we define $\beta \in \mathbf{N}^d$ by $\beta_j = b$, if $0 \leqslant j \leqslant v$, and β_j is any natural number greater than one, and $r \in \mathbf{2N}^d$ by that $r_j = [\theta\alpha_1] + b + 1$, if $0 \leqslant j \leqslant v$, and r_j is any even number such that $(r_j - \beta_j)/\alpha_j > (r_1 - \beta_1)/\alpha_1$ and $(r_j - \beta_j)/\alpha_j \geqslant (r_i - \beta_i)/\alpha_i$ for any $v < i < j \leqslant d$ if $v < j \leqslant d$. Obviously, the set $G = \{(\alpha, r, \beta)\}$ satisfies Conditions (i)–(iv) in Theorem 1.

Let us first prove the direct and inverse inequalities (3)–(4) for the case where $A = \{\alpha\}$. Suppose that $G = \{\gamma = (\alpha, r, \beta)\}$ is any set satisfying Conditions (i)–(iv) in Theorem 1. For a nonnegative integer n, we define the operators

$$T_n f := \sum_{s \in U_n} \delta_s f, \qquad S_n f := \sum_{m=0}^{n} T_m f$$

for functions $f \in L_p(\mathbf{T}^d)$, where

$$U_0 := \left\{ s \in \mathbf{Z}^d_+ : \prod_{j \in J_\alpha} s_j = 0 \right\},$$

$$U_n := \left\{ s \in \mathbf{Z}^d_+ : \prod_{j \in J_\alpha} s_j \neq 0;\ n - 1 \leqslant \langle \alpha, s \rangle < n \right\}, \qquad n \geqslant 1.$$

From Theorem 3 it follows that $f \in L_p(\mathbf{T}^d)$ can be represented as the series

$$f = \sum_{n=0}^{\infty} T_n f$$

converging in the sense of $L_p(\mathbf{T}^d)$. Moreover, Lemma 1 gives

$$\|f\|_p \ll \left(\sum_{n=0}^{\infty} \|T_m f\|_p^{p^*} \right)^{1/p^*}.$$

Since $T_m f \in \mathcal{P}(Z(\alpha, 2^n))$ and the operators T_m and D_t^γ commute, we have by Lemma 4 and Theorem 3

$$\begin{aligned} \|T_m f\|_p &\ll \|D^\gamma_{2^{-m}} T_m f\|_p = \|T_m D^\gamma_{2^{-m}} f\|_p \\ &\ll \|D^\gamma_{2^{-m}} f\|_p \leqslant \Omega^\gamma(f, 2^{-m})_p. \end{aligned}$$

Therefore, by the inclusion $S_n f \in \mathcal{P}^{\alpha}_{2^n}$ we obtain the estimates

$$E^{\alpha}_{2^n}(f)_p \leqslant \|f - S_n f\|_p \ll \left(\sum_{m=n+1}^{\infty} \|T_m f\|_p^{p^*} \right)^{1/p^*} \ll \left(\sum_{m=n+1}^{\infty} \{\Omega^{\gamma}(f, 2^{-m})_p\}^{p^*} \right)^{1/p^*}$$

(here and later, if in a notation $A = \{\alpha\}$, the brackets $\{\ \}$ can be dropped). This proves the case $A = \{\alpha\}$ of the direct inequality (3). Next, we will prove this case of the inverse inequality (4). For a nonnegative integer n, we define the operator

$$P_n f := \sum_{s \in V_n} \delta_s f$$

for functions $f \in L_p(\mathbf{T}^d)$, where $V_0 := U_0$ and

$$V_n := \left\{ s \in \mathbf{Z}^d_+ : \prod_{j \in J_\alpha} s_j \neq 0, n + n_0 - 1 \leqslant \langle \alpha, s \rangle < n + n_0 \right\}$$

with $n_0 = \sum_{j \in J_\alpha} \alpha_j$. Similarly to the proof of (3), f can be represented as the series

$$f = \sum_{n=0}^{\infty} P_n f$$

converging in the sense of $L_p(\mathbf{T}^d)$ and, moreover, we have

$$\|f\|_p \ll \left(\sum_{n=0}^{\infty} \|P_n f\|_p^{p^*} \right)^{1/p^*}. \tag{24}$$

It is easy to verify the following inclusion for any nonnegative integer m:

$$\Gamma_\alpha(2^m) \subset \bigcup_{l=0}^{m} \bigcup_{s \in V_l} \square_s.$$

This implies that $\Gamma_\alpha(2^{m-1}) \bigcap \{\bigcup_{s \in U_l} \square_s\} = \varnothing$ for $l \geqslant m$. Hence, we have $\delta_s(f - g) = \delta_s f$ for any $s \in V_l$, $l \geqslant m$, and $g \in \mathcal{P}^{\alpha}_{2^{m-1}}$. Therefore, from Theorem 3 it follows that for any $g \in \mathcal{P}^{\alpha}_{2^{m-1}}$,

$$\|f - g\|_p \gg \left\| \sum_{l=m}^{\infty} \left(\sum_{s \in V_l} |\delta_s(f - g)|^2 \right)^{1/2} \right\|_p = \left\| \left(\sum_{l=m}^{\infty} \sum_{s \in V_l} |\delta_s f|^2 \right)^{1/2} \right\|_p \geqslant \|P_m f\|_p.$$

Thus, we have proved the following inequality for any nonnegative integer m:

$$\|P_m f\|_p \ll E^{\alpha}_{2^m-1}(f)_p. \tag{25}$$

Let $n-1 \leqslant \xi \leqslant n$ for some natural number n. Because the operators D^{γ}_t and P_m commute, from (24) we obtain that

$$\|D^{\gamma}_{2^{-\xi}} f\|_p \ll \left(\sum_{m=1}^{n} \|D^{\gamma}_{2^{-n}} T_m f\|_p^{p^*} \right)^{1/p^*} + \sum_{m=n+1}^{\infty} \|D^{\gamma}_{2^{-n}} T_m f\|_p.$$

The first sum is taken from 1 to n by virtue of identity $D^{\gamma}_t P_0 f \equiv 0$. We have by Lemma 2 and (25) for $m \leqslant n$,

$$\begin{aligned}\|D^{\gamma}_{2^{-\xi}} P_m f\|_p &\ll 2^{-\rho(n-m)}(n-m+1)^{\nu-1} \|P_m f\|_p \\ &\ll 2^{-\rho(n-m)}(n-m+1)^{\nu-1} E^{\alpha}_{2^m-1}(f)_p,\end{aligned}$$

and, similarly, for $m \geqslant n$,

$$\|D^{\gamma}_{2^{-\xi}} P_m f\|_p \ll 2^{-\rho'(m-n)}(m-n)^{\nu'-1} E^{\alpha}_{2^m-1}(f)_p.$$

Therefore,

$$\begin{aligned}\|D^{\gamma}_{2^{-\xi}} f\|_p &\ll 2^{-\rho n} \left(\sum_{m=1}^{n} \{2^{\rho m}(n-m+1)^{\nu-1} E^{\alpha}_{2^m-1}(f)_p\}^{p^*} \right)^{1/p^*} \\ &\quad + E^{\alpha}_{2^n}(f)_p \sum_{m=n+1}^{\infty} 2^{-\rho'(m-n)}(m-n)^{\nu'-1} \\ &\ll 2^{-\rho n} \left(\sum_{m=0}^{n} \{2^{\rho m}(n-m+1)^{\nu-1} E^{\alpha}_{2^m}(f)_p\}^{p^*} \right)^{1/p^*} \ll F_n,\end{aligned} \tag{26}$$

where F_n denotes the right-hand side of (4). By using the nonincreasing property of $E^{\alpha}_{2^m}(f)_p$ with respect to m, it is easy to verify the inequality $F_m \ll F_n$ for all m and n with $m \geqslant n$. This and (26) imply the inequality (4) for the case $A = \{\alpha\}$.

To prove (3)–(4) for the case where A is a finite set, we need the following equivalences for any $f \in L_p(\mathbf{T}^d)$ and $t > 0$:

$$\Omega^G(f, t)_p \asymp \max_{\gamma \in G} \Omega^{\gamma}(f, t)_p, \qquad E^A_t(f)_p \asymp \max_{\alpha \in A} E^{\alpha}_t(f)_p. \tag{27}$$

The first equivalence (27) follows immediately from the definition and the second from Theorem 1.6 in [14].

We are now in position to prove the case where A is a finite set. By (27) and the case $A=\{\alpha\}$, we have for any $f \in L_p(\mathbf{T}^d)$ and any nonnegative integer n

$$\begin{aligned} E^A_{2^n}(f)_p &\asymp \max_{\alpha \in A} E^A_{2^n}(f)_p \\ &\ll \max_{\alpha \in A} \left(\sum_{m=n+1}^{\infty} \{\Omega^\gamma(f, 2^{-m})_p\}^{p^*} \right)^{1/p^*} \\ &\ll \left(\sum_{m=n+1}^{\infty} \{\max_{\gamma \in G} \Omega^\gamma(f, 2^{-m})_p\}^{p^*} \right)^{1/p^*} \\ &\asymp \left(\sum_{m=n+1}^{\infty} \{\Omega^\gamma(f, 2^{-m})_p\}^{p^*} \right)^{1/p^*}. \end{aligned}$$

Thus, the direct inequality (3) in Theorem 1 is proved. The inverse inequality (4) can be proved in a similar way.

Proof of Theorem 2. Theorem 1 implies that

$$E^A_{2n}(f)_p \ll \sum_{m=n+1}^{\infty} \Omega^G(f, 2^{-m})_p,$$

$$\Omega^G(f, 2^{-n})_p \ll 2^{-\rho n} \sum_{m=0}^{n} 2^{\rho m}(n-m+1)^{\nu-1} E^A_{2^m}(f)_p.$$

Hence, by using Lemma 5, Theorem 2 can be proved in a way similar to the proof of Theorem 1.1 in [4].

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